Precession of a Spinning Ball Rolling Down an Inclined Plane

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Precession of a spinning ball rolling down an inclined plane

A routine problem in an introductory physics course considers a rectangular block at rest on a plane inclined at angle α to the horizontal. In order for the block not to slide down the incline, the coefficient of sliding friction, μ, must be at least $\tan\alpha$. The situation is similar for the case of a ball rolling down an inclined plane. In order for a solid ball to roll without slipping down the inclined plane, μ must be at least $(2/7)\tan\alpha$. In both cases, static friction is responsible for the observed effects and one can find treatments of these topics in most introductory physics textbooks. Notice that when $\alpha = 0$, no frictional force is required for the ball to roll at constant speed, just as no frictional force would be required to keep the rectangular block from sliding on a horizontal plane. In the case of a rolling ball that is accelerating, a frictional force acts to produce a torque about the center of mass and, thus, plays an important role in the acceleration of the ball, whether on a horizontal[1,2] or inclined plane.

A more advanced problem is one where a ball is projected horizontally with initial spin about an axis that does not coincide with the rolling axis.[3] The problem is encountered when a bowling ball is projected with spin down a bowling alley, although in that case sliding friction dominates.[4,5] The frictional torque on the ball changes both the spin and the spin axis as the ball proceeds down the alley. From a teaching point of view, it is more convenient to examine or to demonstrate the effects of friction on a spinning ball using a simple inclined plane. For example, suppose that we change the usual starting condition so that a ball is spinning about a vertical axis when it starts to roll down the incline. What happens then? Will the spin axis remain vertical or will the axis precess into a horizontal position? Will the coefficient of friction be affected? The geometry is shown in Figs. 1 and 2.

The dynamics can be described in an XYZ coordinate system where Y is down the plane, Z is perpendicular to the plane and X is across the plane. We assume that the ball is released when spinning about a nearly vertical axis lying in the XZ plane and inclined at angle θ to the Z axis. The spin axis is shown in Fig. 1(b) which is a rear view of the ball as it rolls down the incline. Normally, the ball is released from rest and the spin axis is then horizontal, with $\theta = 90°$. In that case, the ball rolls along a circular path of radius R on the ball's surface. However, if the spin axis is tilted then the ball will roll along a circular path of radius $r < R$

on the ball's surface, located near the bottom of the ball, while it rolls down a straight line path on the surface of the the incline. From the geometry of Fig. 1(b), $r = R\sin\theta$.

Before the ball starts rolling, it may begin by sliding down the incline. For example, if the ball in Fig. 1(b) is given an initial spin without an initial velocity down the incline, then the ball will slide on the surface at the start. Sliding friction will quickly reduce the spin and increase the linear velocity until the ball starts to roll, at which point the friction force decreases substantially and the ball will then roll to the bottom of the incline. In a bowling alley, an initial sliding phase commences because the ball is projected at a relatively large horizontal speed with only a small forward angular velocity.

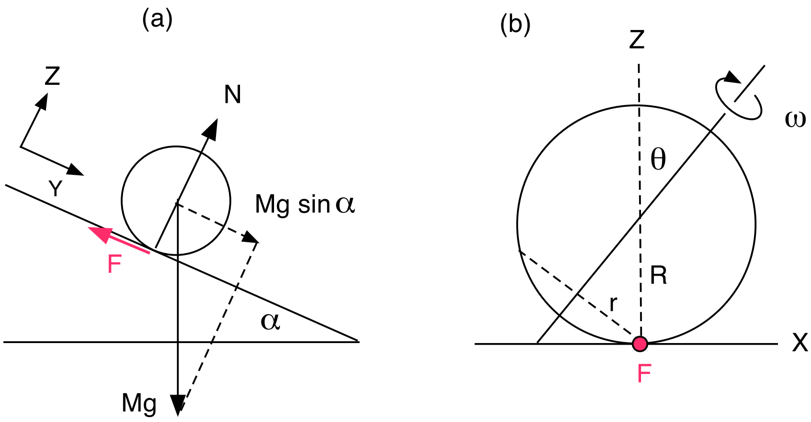

FIG. 1: A ball of radius R rolls in the Y direction down an inclined plane while spinning at angular velocity ω about an axis in the XZ plane inclined at angle θ to the Z axis. F is the frictional force acting on the ball. If the ball is launched without spin then $\theta = 90°$.

The contact point at the bottom of the ball will be instantaneously at rest, and hence the ball will roll down the incline, if $r\,\omega = v$ where ω is the angular velocity of the ball and v is the speed of the center of mass of the ball down the incline. The speed v is determined by the equation

$$M\frac{dv}{dt} = Mg\sin\alpha - F \tag{1}$$

where F is the frictional force and M is the mass of the ball. As shown in Fig. 1(b), F acts at a perpendicular distance r from the spin axis, so the torque about the axis is given by

$$Fr = I_{cm}\frac{d\omega}{dt} \tag{2}$$

where $I_{cm} = 0.4MR^2$ is the moment of inertia of a solid ball. Alternatively,

$$FR = \frac{dL_X}{dt} = \frac{d(I_{cm}\,\omega\sin\theta)}{dt} \tag{3}$$

where FR is the torque about the center of mass, acting in the X direction, and $L_X = I_{cm}\,\omega\sin\theta$ is the X component of the angular momentum. Given that $v = r\,\omega = R\,\omega\sin\theta$, we find from Eq. (3) that

$$F = \frac{I_{cm}}{R^2}\frac{dv}{dt} = 0.4M\frac{dv}{dt} \tag{4}$$

which can be substituted into Eq. (1) to show that the acceleration, a, down the incline is

$$a = \frac{dv}{dt} = \frac{g\sin\alpha}{1.4} \tag{5}$$

We therefore end up with the surprising result that the acceleration of the ball down the incline does not depend on the initial spin of the ball, nor does it depend on the initial inclination of the spin axis. Furthermore, it does not depend on the mass or the radius of the ball. The acceleration is exactly the same as that when the ball has no initial spin, and so is the frictional force and the coefficient of friction. The normal reaction force is given by $N = Mg\cos\alpha$ so the ball will roll without slipping provided that $F/N = (2/7)\tan\alpha$.

How then does the spin vary with time and what happens to the spin axis? The above equations do not provide an answer. A simple answer is obtained by considering the torque about the Z axis. Since no such torque exists, the angular momentum in the Z direction remains constant. That is, $I_{cm}\,\omega\cos\theta$ remains constant, so

$$\cos\theta\,\frac{d\omega}{dt} = \omega\sin\theta\,\frac{d\theta}{dt} \tag{6}$$

Since $v = R\,\omega\sin\theta$, we also have

$$\frac{dv}{dt} = R\sin\theta\,\frac{d\omega}{dt} + R\,\omega\cos\theta\,\frac{d\theta}{dt} \tag{7}$$

Equations (6) and (7) together indicate that

$$v\,\frac{dv}{dt} = R^2\omega\,\frac{d\omega}{dt} \tag{8}$$

which is easily integrated to give

$$v^2 = R^2(\omega^2 - \omega_0^2) \tag{9}$$

where ω_0 is the initial spin. Since v increases with time, ω also increases with time and the angle θ increases with time in such a way that $\omega\cos\theta$ remains constant.

Equation (9) can also be derived by equating the kinetic energy at any point down the incline to the total energy at the top of the incline. At the top, the total energy is $Mgh + 0.5I_{cm}\omega_0^2$ where h is the vertical height of the ball above some arbitrary point at a distance x down the incline, so $h = x\sin\alpha$. At distance x down the incline, $v^2 = 2ax = gx\sin\alpha/0.7$ so $Mgh = 0.7Mv^2$. The energy relation is therefore $0.7Mv^2+0.2MR^2\omega_0^2 = 0.5Mv^2+0.2MR^2\omega^2$ which leads directly to Eq. (9).

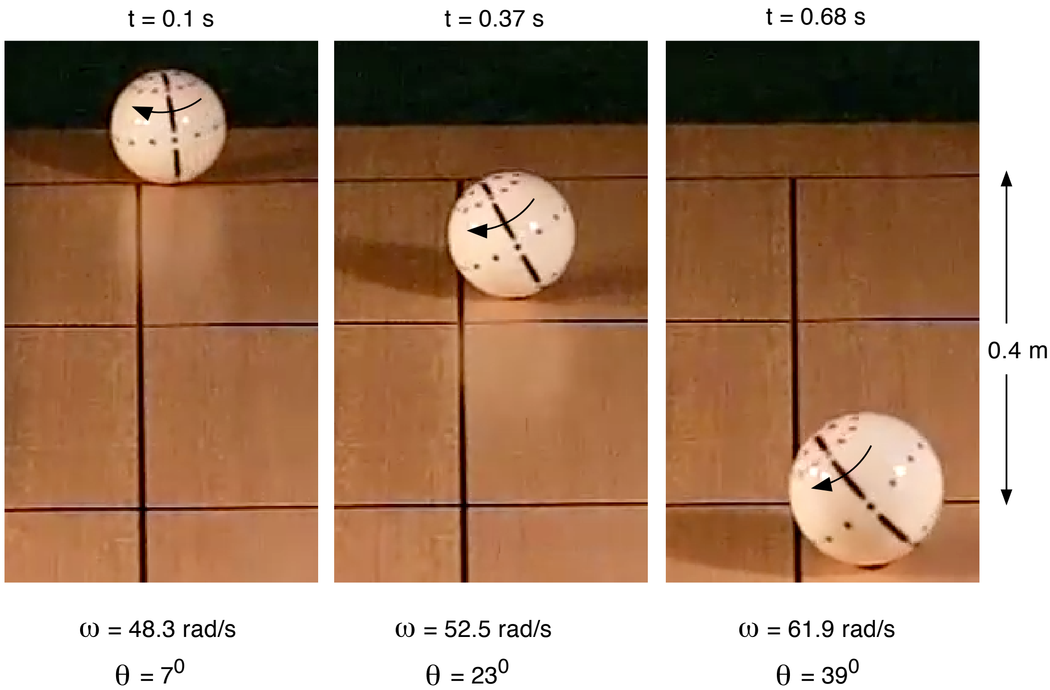

FIG. 2: Results obtained with a billiard ball rolling down an inclined plane when $\alpha = 11.5°$.

The above relations were tested by filming the motion of a 50 mm diameter billiard ball rolling down an incline of length 0.6 m with $\alpha = 11.5°$. The ball was spun by hand about a nearly vertical axis and projected with zero initial speed in the X direction so that it rolled straight down the incline without traversing across the incline. The arrangement and some typical results are shown in Fig. 2. An equator line was drawn around the ball in order to determine both the axis tilt and the time taken to complete each revolution. The rotation angle was plotted as a function of time for eight complete revolutions, and fitted with a cubic to calculate ω vs time. The same data was used to measure the tilt angle of the equator line,

θ, directly from the video image. A different procedure was used to calculate v vs time. In that case, the ball was viewed side-on and the displacement of the ball down the incline was plotted vs time at intervals of 0.02 s using Tracker software. A quadratic fit to the position data was then used to calculate v vs time. Graphs showing v, θ, ω and $\omega \cos \theta$ vs time are shown in Fig. 3. The measured value of θ started at zero when $t = 0$ but the measured value of v was not exactly zero since the time origin was chosen for convenience to start when $\theta = 0$.

Within experimental error, the acceleration of the ball down the incline (1.39 ± 0.01 m/s^2) was consistent with Eq. (5) and was the same as that when the ball had no initial spin. The measured angular velocity was consistent with Eq. (9) to better than 1%, and $\omega \cos \theta$ ($= 47.8 \pm 0.4$) remained constant to within 1%, as indicated by Eq. (6).

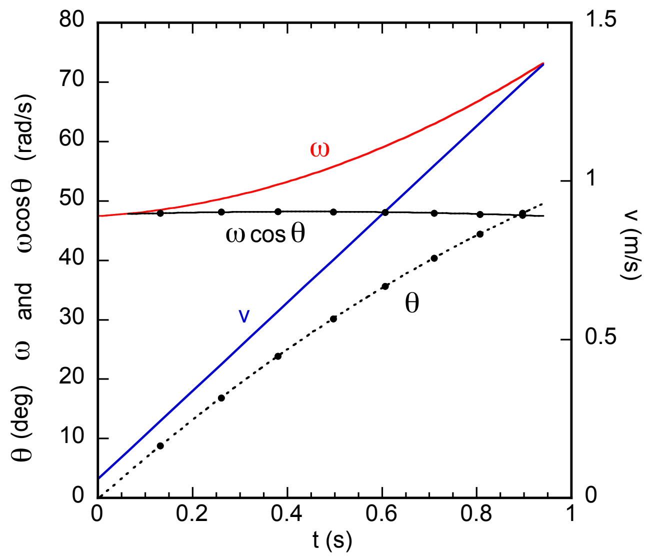

FIG. 3: Measured ball speed, v, (right axis), axis tilt (θ), angular velocity ω, and $\omega \cos \theta$ vs time. Dots indicate θ and $\omega \cos \theta$ data points, one per revolution, and curves are quadratic or linear fits to raw data.

Qualitative features of this experiment are of interest from a teaching point of view. The initial angular momentum in the Z direction is retained as the ball rolls down the incline, with the result that the spin axis retains a strong Z component, despite the natural tendency for the spin axis to rotate into a position parallel to the X axis. The initial direction of the angular momentum vector points down into the inclined plane in Fig. 2, along the $-$Z axis.

The torque vector remains horizontal and points to the right (in the −X direction) when viewed from the front of the ball in Fig. 2. Since the torque equals the rate of change of the angular momentum, the change in the angular momentum points to the right. Consequently, the spin axis rotates counter-clockwise as the ball rolls down the incline. If the ball is spun in the opposite direction, then the axis leans to the right instead of to the left, and it rotates clockwise as the ball rolls down the incline.

Rotation of the spin axis in this manner is closely analogous to the precession of a gyroscope.[6] The torque remains perpendicular to the spin axis in a gyroscope, the spin remains constant and the rate of precession remains constant. In the present case, the spin axis rotates towards the torque vector, as it does in a gyroscope, but its rate of rotation (ie $d\theta/dt$) decreases with time and the spin increases with time. Nevertheless, the initial rate of rotation, when $\theta = 0$, is the same as the gyroscopic precession rate, given from Eq. (3) by $d\theta/dt = FR/I_{cm}\omega$ = torque/angular momentum. In the present experiment, precession appears as a naturally expected result since the ball rolls down the incline, as expected, and the spin axis rotates in the expected direction. With a gyroscope, precession is a counter-intuitive and unexpected result since the weight of the spinning disk is not supported by anything underneath the disk and the disk itself rotates in a horizontal rather than a vertical plane.

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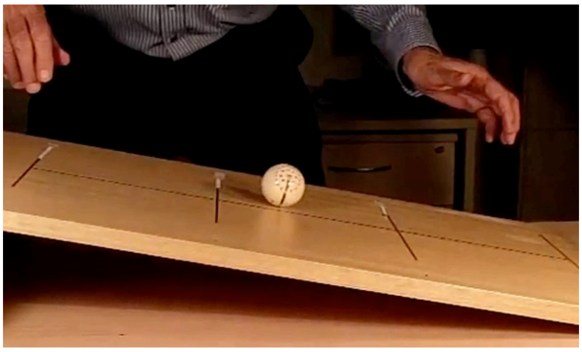
Supplementary videos



Side View

Front View